Direct Evaluation of Withdrawal Equations

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It has been our interest to predict entrainment in withdrawal. Although several general equations have appeared, numerical evaluation has posed a difficulty. The problem of evaluation has been overcome by iterative methods but special precautions are needed in some regions. However, Ellis fluid equations have not been evaluated.

The purpose of this paper is to describe a direct method for evaluating complex withdrawal equations without iteration. This method was developed for Ellis fluids.

The equations concerned are those for predicting film thickness (h) as a function of withdrawal speed (u), fluid properties (such as density, surface tension, and viscosity), and other conditions. We will restrict the discussion to the steady and continuous upward passage of a symmetrical object through the free surface of a liquid bath.

AN INTRODUCTION TO THE DIRECT METHOD

One prediction of film thickness is the gravity corrected theory for withdrawal of flat plates from Newtonian fluids (*2*)

$$h\left(\frac{\rho g}{\mu u}\right)^{1/2} = 0.944\left(\frac{\mu u}{\sigma}\right)^{1/6}\left[1-\frac{h^2\rho g}{\mu u}\right]^{2/3} \tag{1}$$

Equation (1) has been verified experimentally for $\mu u/\sigma < 1$. This result was originally presented in nondimensional form using the speed parameter $C_o \equiv \mu u/\sigma$ and the thickness parameter $T_o \equiv h(\rho g/\mu u)^{1/2}$

$$T_o = 0.944\ C_o^{1/6}(1-T_o^2)^{2/3} \tag{2}$$

Equation (2) is relatively easy to solve for T_o at given values of C_o using standard iterative techniques; thus the direct method is not particularly helpful here. We chose this equation as a simple illustration of the direct method.

The development of the direct method for this case is shown by using another nondimensional thickness parameter $D \equiv h(\rho g/\sigma)^{1/2}$. Thus

$$D = 0.944\ (C_o - D^2)^{2/3}$$

To obtain C_o explicitly, we rearrange to the desired result as:

$$C_o = 1.090\ D^{3/2} + D^2 \tag{3}$$

The dimensional form of Equation (3) is

$$u = 1.09\left(\frac{\sigma}{\rho g}\right)^{1/4}\frac{\rho g h^{3/2}}{\mu} + \frac{\rho g h^2}{\mu} \tag{4}$$

To use Equation (4) for a given fluid, one can explicitly solve the velocity needed to provide a specified film thickness. The explicit speed form has the usual advantages compared to the implicit form: the explicit form is easier to use for predicting the behavior of a fluid and for preparing a plot to compare with data. If desired, values of film thickness at a given speed may be obtained by interpolation.

A special advantage of the explicit form in withdrawal is that each term has a more readily understood physical significance, and may be directly compared with related equations.

ELLIS FLUIDS (FLAT PLATES)

The benefit in eliminating iteration and in obtaining forms easier to understand is greater for complex equations. Some of the most complex expressions are those for Non-Newtonian fluids, such as those for an Ellis fluid.

$$-\frac{dv}{dy} = (a_o + a_1{}^{\alpha}\,|\tau|^{\alpha-1})\tau$$

The theory for flat plate withdrawal from an Ellis fluid (*1*) is

$$D = 0.944\left[\frac{C_o(1-T_o^2-T_1{}^{\alpha+1})}{1+T_2}\right]^{2/3} \tag{5}$$

where

$$T_1 \equiv h\,[(\rho g a_1)^{\alpha}/u]^{1/(\alpha+1)}$$

$$T_2 = [3\alpha/(\alpha+2)]\,[C_o/C_1]\,T_1{}^{(\alpha^2-1)/\alpha}$$

$$C_1 = (h^{\alpha-1}u)^{1/\alpha}/(a_1\sigma)$$

To use low shear ($1/a_o$) in place of viscosity, C_o and T_o are rewritten as $C_o \equiv u/a_o\sigma$ and $T_o \equiv h(\rho g\ a_o/u)^{1/2}$. The definition of C_1 is a correction of an earlier typographical error (*1*). Equation (5) has not been evaluated numerically.

Following the earlier development Equation (5) can be arranged to the desired result

$$u = 1.09\ (\rho g a_o)\left(\frac{\sigma}{\rho g}\right)^{1/4} h^{3/2} + (\rho g a_o)h^2 + \frac{1.09(3\alpha)}{(\alpha+2)}(\rho g a_1)^{\alpha}\left(\frac{\sigma}{\rho g}\right)^{1/4} h^{\alpha+(1/2)} + (\rho g a_1)^{\alpha}\ h^{\alpha+1} \tag{6}$$

One nondimensional form of Equation (6) is

$$C_o = D^2\left[\frac{1.09}{D^{1/2}}+1\right] + D^{2\alpha}A\left[\frac{1.09(3\alpha)}{(\alpha+2)D^{1/2}}+1\right] \tag{7}$$

where

$$A = \frac{(a_1\sigma/h)^{\alpha}}{(a_o\sigma/h)}$$

There are several further benefits in expressing the Ellis theory in the speed explicit form. The polynomial form of the four term equation permits calculation of each term's relative contribution, and permits the use of standard iteration methods for determining values of thickness at specified values of speed. The analytical form of Equation (6) is directly comparable with related equations for free drainage and power law withdrawal. The comparisons were sought earlier by using the implicit form Equation (5) but none were found (*1*).

CYLINDERS (NEWTONIAN FLUIDS)

Another complex expression is the gravity corrected theory for withdrawal of radius R cylinders (*3*)

$$T_o M = 0.944\ C_o^{1/6}\left[1-\frac{2\ S^2\ G^2\ Z}{C_o}\right]^{2/3} \tag{8}$$

where

$$G \equiv R\ (\rho g/2\sigma)^{1/2}$$

$$S \equiv (h+R)/R$$

$$M = M(G, S) \equiv \frac{2.4\ G^{0.85}\ S^{0.85}}{1 + 2.4\ G^{0.85}\ S^{0.85}} + \frac{0.5}{GS}$$

$$Z = Z(S) \equiv \ln S - \frac{1}{2}\left(\frac{S^2-1}{S^2}\right)$$

Equation (8) has been evaluated for certain specified values of C_o and G by solving an iteration method but special precautions were necessary. Following the earlier procedure, Equation (8) can be written in D notation and rearranged to be explicit in C_o

$$D\ M = 0.944\ (C_o - 2\ S^2G^2Z)^{2/3}$$

$$C_o = 1.09\ (D\ M)^{3/2} + 2\ S^2G^2Z \qquad (9)$$

The speed explicit form is

$$u = 1.09\left(\frac{\sigma}{\rho g}\right)^{1/4}\frac{\rho g(hM)^{3/2}}{\mu} + \frac{\rho g\ Z\ (h+R)^2}{\mu} \qquad (10)$$

Given radius, fluid properties, and film thickness, one can calculate the speed by Equation (10) without iteration.

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Crystallization on a Constant Temperature Surface

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In a recent article (*1*) Harriott has investigated the effect of heat and mass transfer on ice crystal growth rates for crystallization in the bulk liquid of a subcooled water and salt solution. He concluded from experiment and theory that heat and mass transfer do affect the crystal growth rates, but that there could be an appreciable crystallization resistance especially at low subcooling (0.01 to 0.02°C.).

This paper presents a similar theoretical analysis (for a two component solution) of the heat and mass transfer effect on the rate of crystallization on a constant temperature surface. This case is of considerable practical importance. It is especially applicable to scraped pipe crystallizers (2) which are used extensively in the chemical process industries.

Crystallization on a cold surface involves conducting the latent and sensible heat through the solid phase to the cold surface.

The following important assumptions will be made:

1. The thermal capacity of the solid phase is negligible, that is, the temperature in the solid phase is linear.
2. There is no driving force required for crystallization, as the interfacial temperature is the true freezing point of the solution.
3. The interfacial heat and mass transfer coefficients are constant.
4. The freezing point relationship can be expressed as the following linear relationship:

$$c = A - B(T) \qquad (1)$$

5. The bulk of the liquid is at saturation temperature (realistic for continuous flow devices).
6. The bulk flow contribution to heat and mass transfer is negligible.

The growth rate of the solid phase from a heat balance is

$$k_s\frac{(T_i - T_w)}{x} = \lambda\rho_s\frac{dx}{dt} + h(T_b - T_i) \qquad (2)$$

A mass balance on the component diffusing away from the interface is given by Harriott (*1*):

$$K(c_i - c_b) = \frac{c_b}{\rho_b - c_b}\rho_s\frac{dx}{dt} \qquad (3)$$

Substituting for $(c_i - c_b)$ in Equation (3) we obtain from Equation (1):

$$T_b - T_i = \frac{c_b}{\rho_b - c_b}\frac{\rho_s}{KB}\frac{dx}{dt} \qquad (4)$$

Equation (2) can be rearranged as follows:

$$\frac{k_s}{x}[(T_b - T_w) - (T_b - T_i)] = \lambda\rho_s\frac{dx}{dt} - h(T_b - T_i) \qquad (5)$$

Substituting $(T_b - T_i)$ from Equation (4) into Equation (5), simplifying and integrating with the initial condition $x = 0$ at $t = 0$ we obtain

$$x = \sqrt{\alpha^2 + 2\beta t} - \alpha \qquad (6)$$

where α, β and ϕ (same as Harriott's ϕ) are as follows:

$$\alpha = \frac{k_s\phi}{h(1+\phi)} \qquad (7)$$

$$\beta = \frac{k_s(T_b - T_w)}{\lambda\rho_s(1+\phi)} \qquad (8)$$

$$\phi = \frac{hc_b}{\lambda K(\rho_b - c_b)B} = \frac{k_b N_{Nu} c_b}{\lambda D N_{Sh}(\rho_b - c_b)B} \qquad (9)$$

Various analogies between heat and mass transfer (3) predict that

$$\frac{N_{Nu}}{N_{Sh}} = \left(\frac{N_{Pr}}{N_{Sc}}\right)^a \qquad (10)$$

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